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Differential equations, Frobenius theorem and local flows on supermanifolds†

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Abstract. The classical Frobenius theorem, both in its local and global formulations, is generalised to superanalytic supermanifolds. As an application, it is proved that a coset space G/H (where both G and H are super Lie groups) is a supermanifold. Existence and uniqueness of local flows of tangent vector fields is proved.

1. Introduction

The theorem of Frobenius is a fundamental tool for the development of ordinary differential geometry. It is therefore no surprise that in the study of the geometry of supermanifolds (Rogers 1980, 1981, Jadczyk and Pilch 1981, Cianci 1984, Bruzzo and Cianci 1984a, b) Frobenius theorem is also found to be of basic importance. For instance, it is essential to the study of the geometry of coset spaces G/H (both G and H being super Lie groups), which are involved in the symmetry breaking of supergauge theories and in the dimensional reduction of field theories on supermanifolds as well as in the (super) group manifold approach to gauge theories (Ne'eman and Regge 1978).

Proofs of the local Frobenius theorem for Kostant's graded manifolds, as well as for the strictly related Berezin-Leites supermanifolds, have appeared in the literature: see Kac and Koronkievich (1971), Giacchetti and Ricci (1981), Shander (1983b). In this paper we generalise these results in two ways: firstly, we follow the more general axiomatic approach to supermanifolds by Jadczyk and Pilch; secondly, we also prove the global version of the theorem.

It is our opinion that the axiomatic approach is more profitable, since it allows us to dispose of a specific choice of the Banach space used to model the supermanifold, and seems more suitable for quantum-mechanical applications (Jadczyk and Pilch 1983). A further advantage is that the resulting differential geometry is similar to the ordinary theory and more flexible for physical applications (Bruzzo and Cianci 1984a, b, c).

We prove the theorem in the case of superanalytic supermanifolds. Since the local theorem is basically a matter of integration of a system of differential equations, we have to prove an existence and uniqueness theorem for a certain type of differential equation where both the dependent and the independent variables take values in

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vector superspaces. Once again, the result is obtained by resorting to the underlying Banach structure.

The paper is arranged as follows. In § 2 we state the main results; then, the Frobenius theorem is applied to the study of coset supermanifolds. A theorem on the existence and uniqueness of local flows of tangent vector fields is proved. This requires a redefinition of the tangent bundle, and the new definition is compared with the old one. Section 3 is devoted to the proofs.

Concerning definitions and notations, we rely completely on Jadczyk and Pilch (1981) and Bruzzo and Cianci (1984a). The necessary background in Banach space analysis and infinite-dimensional differential geometry may be found in Dieudonné (1960) and Lang (1972).

2. Frobenius theorem and local flows

Let M be an (m, n)-dimensional supermanifold, and $\mathbb{E}$ a *vector superspace* (VSS). A *vector superbundle* (VSB) E over M with standard fibre $\mathbb{E}$ is defined as in the case of ordinary manifolds (Lang 1972), the basic principle being here the supersmoothness of the maps involved. The projection $E \to M$ will be denoted by π_E.

A *section* of E is a supersmooth map $s: U \to E$ (U is an open in M) such that $\pi_E \circ s = id_U$. Quite obviously, $T(M)$—defined as the union of tangent spaces to M—is a VSB with standard fibre $Q^{m,n}$, and its sections are the tangent vector fields on M.

A *sub-bundle* of E is a triple (F, f, f'), where F is a VSB over M, $f: F \to E$ and $f': M \to M$ are supersmooth, and

(i) $\pi_E \circ f = f' \circ \pi_F$;

(ii) $f: \pi_F^{-1}(x) \to \pi_E^{-1} \circ f'(x)$ is a vss homomorphism.

A sub-bundle F of $T(M)$ is said to be *involutive* if, given sections X, Y of F, $[X, Y]$ is a section of F too. F is said to be *integrable* if for each $x \in M$ there exists a sub-supermanifold (N, i) of M such that $x \in i(N)$ and $i_* T_y(N) = \pi_F^{-1} \circ i(y)$ for each $y \in N$. Such a sub-supermanifold is called an *integral S-manifold* (ISM) of F through x.

Now we are in position to state the local Frobenius theorem.

Theorem 2.1. Let M be an (m, n)-dimensional superanalytic (SA) manifold. A sub-bundle F of $T(M)$ is integrable if and only if it is involutive. If that is the case, around every $x \in M$ there is a chart $(U, x^1(\cdot), \ldots, x^{m+n}(\cdot))$ such that the S-manifolds given by $x^{p+1}, \ldots, x^m, x^{m+q}, \ldots, x^{m+n} = \text{const}$ are ISM's of F (here $\dim \mathbb{F} = (p, q)$). Finally, the ISM's of F are locally unique, in the sense that, given two ISM's (N, i) and (K, j) of F through x, there exists an open neighbourhood V of x such that $i(N) \cap V = j(K) \cap V$.

A *maximal integral S-manifold* of F through x is an ISM which contains any other ISM through x.

Theorem 2.2. (Global Frobenius theorem) Under the same hypotheses of theorem 2.1, there is a unique maximal connected ISM through each $x \in M$.

An interesting straightforward consequence of these results is:

Theorem 2.3. Let G be a super Lie group and (H, i) a closed sub-SL-group of G; let $\pi: G \to G/H$. Then G/H has a unique superanalytic structure such that π is superanalytic. Moreover, there exists local superanalytic sections of G/H in G.

Now we turn to the study of local flows of tangent vector fields. In order to do that, we must look at the structure of $T(M)$ in some detail.

Each tangent space $T_x(M)$ may be defined as the set of equivalence classes of ordinary curves $\gamma: I \to M$ (I being an open neighbourhood of $0 \in \mathbb{R}$) under the usual equivalence relation (Jadczyk and Pilch 1981). One can also introduce a *supercurve* as a supersmooth map $\xi: J \to M$ (J being an open neighbourhood of $0 \in Q_0$) and state that two supercurves ξ, ξ' are equivalent ($\xi \sim \xi'$) if $\xi(0) = \xi'(0) = x \in M$ and $(D\xi^A)_0 = (D\xi'^A)_0$ (in terms of a chart around x; D is the Fréchet differential). Then we have:

Theorem 2.4. The set of equivalence classes of supercurves through $x \in M$ under $\sim$ is vss-isomorphic to $T_x(M)$.

It is straightforward to verify that the isomorphism extends to the corresponding vector superbundles. We may therefore speak of *integral supercurves* of a given tangent vector field.

We may also introduce the related concept of *local flow*, defined as a supersmooth map $\phi: J \times U \to M$, U an open in M, such that:

(i) for each $a \in J$, $\phi(a, \cdot): U \to M$ is a superdiffeomorphism;

(ii) $\phi(0, x) = x$ for each $x \in U$;

(iii) $\phi(a+b, x) = \phi(a, \phi(b, x))$ whenever the two sides are defined.

Using local coordinates, we may associate a tangent vector field X to ϕ:

$$X(p) = [D_a x^A \circ \phi(a, p)]_{a=0} (\partial/\partial x^A)_p, \qquad p \in M.$$

In the SA case there is also a local converse; let us say that a tangent vector $Y \in T(M)$ has *vanishing body* if its components have vanishing body.

Theorem 2.5. Let M be an SA manifold, $X \in T(M)$ an SA field, $x \in M$ and $X(x) \neq 0$. Then there exists an open U around x, an open neighbourhood J of $0 \in Q_0$ and a unique flow $\phi: J \times U \to M$ whose associated vector field is just $X|_U$; moreover ϕ is SA†.

If X has non-vanishing body, this theorem follows from theorem 2.1, since X is in a one-to-one correspondence with a sub-bundle of $T(M)$. If that is not the case, we use the more general result given by the following lemma:

Lemma 2.1. Let U ben an open in $Q^{m,n}$, J an open around $0 \in Q_0, f: J \times U \to Q^{m,n}$ an SA map, and $u_0 \in U$. Then there exists an open ball $\mathscr{C} \subset J$ of centre 0 and a unique mapping $u: \mathscr{C} \to U$ which satisfies the conditions:

$$u'(s) = f(s, u(s)) \tag{2.1a}$$

$$u(0) = u_0; \tag{2.1b}$$

moreover, the dependence of u upon s and u_0 is superanalytic.

† Analogous results in the case of Berezin-Leites supermanifolds, graded manifolds and Rogers' supermanifolds were given respectively by Shander (1980, 1983a), Giacchetti and Ricci (1981) and Boyer and Gitler (1983).

Proof. Take into account equations (2.1) by considering a real s and use the standard local existence and uniqueness theorem (Penot 1970); then apply the z-extension (Rogers 1980, Jadczyk and Pilch 1981) to the solution, hence obtaining an SA function which satisfies all the requirements of the lemma.

An immediate consequence of these results is the following:

Corollary 2.1. Let M be an SA manifold, $X \in T(M)$ superanalytic, $x \in M$. A chart $(U, x^1(\cdot), \ldots, x^{m+n}(\cdot))$ around x such that $X|_U = (\partial/\partial x^1)$ (x^1 even) exists if and only if X has non-vanishing body.

Proof. Since $\mathrm{GL}(m, n)$ acts transitively on the set of elements of $Q^{m,n}$ having non-vanishing body (see theorem 2.4 of Bruzzo and Cianci 1984a), we may choose around x coordinates $y^1(\cdot), \ldots, y^{m+n}(\cdot)$ such that $X(x) = (\partial/\partial y^1)_x$ if and only if $X(x)$ has non-vanishing body. Then the proof goes as in the classical case (see e.g. Warner 1971).

In analogy with this treatment of tangent vector fields, one could envisage defining the odd vector fields on M as equivalence classes of 'odd' supercurves $\xi: Q_1 \to M$. This is not convenient—at least in the SA case—since an SA function of an odd argument is necessarily linear, and this would force the fields to be 'geodesic'.

Finally, let us note that the previous results, while yielding existence and uniqueness properties of solutions of differential equations on vss's do not provide any method to find out the solution explicitly. In the SA case, a useful method is provided by the comparison of power series. For instance, set $M = Q^{1,1}$ and consider the vector field:

$$X = (\partial/\partial x) + \xi(\partial/\partial \xi)$$

where (x, ξ) are coordinates on $Q^{1,1}$. We wish to find the integral curve of X through the point $(0, a)$. We must solve:

$$Dx = 1, \qquad x(0) = 0$$

$$D\xi = x(s)\xi(s), \qquad \xi(0) = a$$

where s is a variable in Q_0. The first equation gives $x(s) = s$; then, setting

$$\xi(s) = \sum_{n=0}^{\infty} b_n s^n \qquad (\text{with } b_0 = a),$$

and substituting into the second equation, we obtain the following relations:

$$b_{2n+1} = 0, \qquad b_{2n} = b_{2n-2}/2n \qquad (n > 0)$$

so that

$$\xi(s) = a \sum_{n=0}^{\infty} \frac{s^{2n}}{(2n)!!}.$$

The integral curve is therefore

$$\gamma(s) = s + a \sum_{n=0}^{\infty} \frac{s^{2n}}{(2n)!!}.$$

3. Proofs

In this section we first prove theorem 2.1, which guarantees the existence of local ISM's; then these are patched together by means of an SA atlas, thus obtaining the maximal ISM through a given point. As a consequence theorem 2.2 may be proved.

Concerning theorem 2.1, one could think of mimicking the classical proof (see e.g. Warner 1971) based on induction on the dimension of the sub-bundle. This does not appear to be useful, since it entails using corollary 2.1 for each of the vector fields which generates the sub-bundle, and this imposes restrictions (for an analogous situation in the case of graded manifolds see Giacchetti and Ricci 1981). It is most profitable to take on the Frobenius theorem in Banach spaces (Penot 1970) and to prove directly the further properties needed in our case (basically, the superanalyticity of some maps).

Thus, let us consider a sub-bundle $\pi_F: F \to M$ of $T(M)$. The standard fibre $\mathbb{F}$ of F is VSS—isomorphic to $Q^{p,q}$, with $p \leq m$, $q \leq n$; moreover, there exists a VSS $\mathbb{G}$ such that $Q^{m,n} \approx \mathbb{F} \oplus \mathbb{G}$ (see theorem 2.4 of Bruzzo and Cianci 1984a). Let us denote by p_1 (resp. p_2) the projecion of $Q^{m,n}$ onto $\mathbb{F}$ (resp. $\mathbb{G}$). Let (U, ψ) be a chart around $x \in M$; we may set $\psi(U) = A \times B$, with $A \subset \mathbb{F}$, $B \subset \mathbb{G}$. A vector $Z_u \in T_u(M)$ has a coordinate representation:

$$Z_u = (X_{(x,y)}, Y_{(x,y)}) \tag{3.1}$$

where $x = p_1\psi(u)$, and $y = p_2\psi(u)$, $X = p_1 Z$, $Y = p_2 Z$. Specialising equation (3.1) to the vector fields V in F (or, more exactly to the sections of $f_*\pi_F^{-1}(U)$, where $f: F \to T(M)$) we have:

$$V_{(x,y)} = (X_{(x,y)}, X_{(x,y)}\phi(x, y)) \tag{3.2}$$

where, for each (x, y), $\phi(x, y)$ is a Q_0-linear map $\mathbb{F} \to \mathbb{G}$; actually, locally we have $f_* = (id, \phi)$, so that $\phi(x, y)$ is Q_0-linear, continuous and SA in x, y; it will be regarded as a matrix, which we assume to act from the right. The involutivity of F may be stated as a condition on ϕ:

$$(s_1, s_2)D_1\phi + (s_1, s_2)D_2\phi = (s_2, s_1)D_1\phi + (s_2, s_1)D_2\phi; \qquad \forall s_1, s_1, s_2 \in F \tag{3.3}$$

where D_1 (resp. D_2) is the Fréchet differential with respect to the first (the second) variable. Equation (3.3) may be written in component notation as follows: if A, B (resp. i, j resp. α, β) are indices in $Q^{m,n}$ (resp. $\mathbb{F}$, resp. $\mathbb{G}$), we have:

$$\begin{gathered} V^A(x, y) = (X^i{}_{(x,y)}, X^j{}_{(x,y)}\phi(x, y)_j{}^\mu) \\ \phi_i{}^\mu{}_{,j} - (-1)^{ij}\phi_j{}^\mu{}_{,i} + \phi_i{}^\alpha\phi_j{}^\mu{}_{,\alpha} - (-1)^{ij}\phi_j{}^\alpha\phi_i{}^\mu{}_{,\alpha} = 0 \end{gathered} \tag{3.4}$$

where commas denote Fréchet differentiation.

Now, from (3.2) we see that, in order to prove our result, we must show that there exists an SA map $\alpha: A \times B \to \mathbb{G}$ such that:

$$\begin{aligned} &D_1\alpha(x, y) = \phi(x, \alpha(x, y)) && \forall (x, y) \in A \times B \\ &\alpha(x_0, y) = y && \text{(for a fixed } x_0 \in A \text{ and each } y \in B\text{).} \end{aligned} \tag{3.5}$$

Looking at $\mathbb{F}$ and $\mathbb{G}$ as B-spaces, we may use theorem 1.1 of Penot (1970) to deduce that an α solution of (3.6) exists if and only if (3.3) holds; furthermore, α is *unique* and *analytic.* Equation (3.5) implies directly that α is SA in x since $D_1\alpha$ is ϕ and therefore acts as a matrix. To prove superanalyticity in y it is enough to look at the Taylor expansion of α around x_0:

$$\alpha(x, y) = y + (x - x_0)\phi(x_0, y) + \tfrac{1}{2}\{(x - x_0, x - x_0)D_1\phi(x_0, y) + [(x - x_0)\phi(x_0, y), x - x_0]D_2\phi(x_0, y)\} + \ldots . \tag{3.6}$$

Each term of this series is SA in y since ϕ is. The absolute convergence of the power series implies that α is SA in y too.

Now consider the maps $\chi_y : x \to (x, \alpha(x, y))$; they are SA and injective, since $D\chi_y$ is not singular for any $(x, y) \in A \times B$. With a suitable restriction of the neighbourhoods involved, and using theorem 2.1 of Penot (1970) and proposition 5.3 of Jadczyk and Pilch (1981), we deduce that χ_y^{-1} exists and is SA. Then the SA diffeomorphism Φ, defined by its coordinate representation $\Phi(x, y) = \chi_y(x)$, maps $A \times \{y\}$ into a local ISM of F and entails the existence of the 'adapted' coordinate system mentioned in the statement of the theorem. The local uniqueness is implicit in the proof.

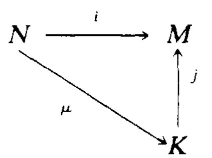
So far we have proved theorem 2.1 in the case of SA manifolds; the case of supersmooth manifolds still presents some problems, since we do not know how to prove that α is supersmooth in y. However, this problem does not arise when we want to find the integral S-curves of a S-smooth tangent vector field through a given initial point, since in that case smoothness with respect to the initial datum is not required.

We are now in a position to prove theorem 2.2. The maximal connected ISM through a given point $p \in M$ may be built up as in the classical case (see e.g. Warner 1971). Let K_p be the totality of the points of M which may be joined to p by means of a $\mathscr{C}^0$, piecewise SA supercurve whose tangent vector lies in F. If $\{(U_\alpha, \psi_\alpha)\}$ is an atlas on M, and $p \in U_\alpha$, set $V_{\alpha p} = U_\alpha \cap K_p$. If the atlas is suitably chosen, $V_{\alpha p}$ is a local ISM of F through x. Define a map $\chi_{\alpha p} : V_p \to \mathbb{F}$ by setting $\chi_{\alpha p}(z) = p_1 \circ \psi_\alpha(z)$. We may show that $\{(V_{\alpha p}, \chi_{\alpha p})\}$ is an SA atlas on K_p. The only non-trivial point in doing that is to check that the transition functions are SA; to this end, let $z \in V_{\alpha p} \cap V_{\beta p}$; then:

$$\chi_{\alpha p} = p_1 \circ \psi_\beta(z) = p_1 \circ (\psi_\beta \circ \psi_\alpha^{-1}) \circ \psi_\alpha(z) = p_1 \circ (\psi_\beta \circ \psi_\alpha^{-1}) \circ (\chi_{\alpha p}(z) + p_2 \circ \psi_\alpha(z)).$$

Since $p_2 \circ \psi_\alpha(z)$ is constant on $V_{\alpha p}$, and $p_1 \circ \psi_\beta \circ \psi_\alpha^{-1}$ is SA, the transition function relating $\chi_{\alpha p}(z)$ to $\chi_{\beta p}(z)$ is SA.

So we have proved that K_p is an SA-manifold. Its maximality is obvious by construction. We have only to prove its uniqueness (up to S-diffeomorphisms). Let (N, i) and (K, j) be two maximal ISM's through p. We have the diagram:



where the maps i, j are SA and injective. N and K being also B-manifolds, we may advocate the uniqueness theorem in B-manifold theory (Lang 1972) to deduce that a $\mathscr{C}^\omega$ diffeomorphism $\mu : N \to K$ exists. In particular, both μ and μ^{-1} are continuous, and a known result (Bruzzo and Cianci 1984a, lemma 3.1) implies that both are SA. So K and N are SA-diffeomorphic, and this concludes the proof.

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